Non-Fragile Output-Feedback Control with Time-Varying Input Delay for Master-Slave Synchronization of Uncertain Fractional-Order Lur'e Systems

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Abstract: **This paper proposes a non-fragile output-feedback control strategy to achieve master-slave synchronization of uncertain fractional-order Lur'e systems (FOLSs) subject to time-varying input delays. The proposed controller guarantees asymptotic convergence of the synchronization error to zero, even in the presence of parametric uncertainties in the system and gain variations in the controller implementation. Based on fractional-order stability criteria, sufficient conditions for the existence of such a controller are derived in terms of linear matrix inequalities (LMIs). The effectiveness and practical applicability of the proposed approach are demonstrated through numerical simulations performed on an uncertain chaotic neural networks, the results of which confirm its robustness.**

Keywords: **Fractional-order systems, Master-slave synchronization, Non-fragile control, Output-feedback control, Time-varying input delay**

I Introduction

Fractional calculus is an important branch of mathematical analysis that extends classical derivatives and integrals from integer-order to non-integer orders. This generalization makes it particularly well-suited for modeling complex systems characterized by memory effects, hereditary properties, and nonlocal interactions [1]–[3]. In contrast to integer-order calculus, which captures only instantaneous rates of change, fractional-order (FO) operators inherently retain past information, thus enabling a more refined and comprehensive representation of a system's historical dependencies. This distinctive feature has rendered FO models increasingly valuable in areas such as electrochemical processes, image processing, thermal diffusion, biological tissues, and networked control systems [4]–[7]. Over the past two decades, numerous physical, biological, and engineering systems have been shown to inherently exhibit FO dynamics, spurring extensive research into their analysis and control [8, 9].

Among the various problems associated with FO systems, the synchronization of chaotic FO systems has attracted considerable attention, largely driven by its potential applications in secure communications, image encryption, information processing, and biomedical engineering [10]–[12]. Chaotic systems are characterized by extreme sensitivity to initial conditions and parameters, such that small perturbations can lead to widely diverging outcomes, rendering long-term prediction generally impossible [13]. Despite this inherent unpredictability, chaos has found extensive applications across numerous real-world phenomena.

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The growing interest in FO chaotic systems stems from the fact that FO models describe many real-world problems more accurately than their integer-order counterparts [14]. However, the practical deployment of such systems requires reliable synchronization schemes that can ensure the slave system faithfully tracks the master system.

It is well recognized that a broad class of chaotic nonlinear systems can be represented in Lur'e form, which comprises a linear forward path and a sector-bounded feedback nonlinearity. This representation encompasses numerous well-known chaotic systems, including Chua's circuits, n-scroll attractors, cellular neural networks, and hyperchaotic oscillators [15]–[18]. Consequently, the synchronization of chaotic Lur'e systems has become a central research topic [19].

In practical control applications, two additional challenges must be addressed: time delays and uncertainties. Time delays are ubiquitous in real-world systems and may arise from communication networks, signal transmission, computational lags, or actuator dynamics, potentially degrading performance or even destabilizing the system [20]. Similarly, parameter uncertainties stemming from modeling inaccuracies and environmental variations can severely impair synchronization performance [21]–[23]. A further, frequently neglected concern is the fragility inherent in controller implementation. Gain variations in the controller– caused by finite word length in digital hardware, rounding errors, or component aging– can lead to performance degradation or instability. To counter this, non-fragile control seeks to design controllers that maintain robustness against such variations [24, 25]. In practice, however, not all system states are available for direct measurement. To overcome this limitation, the present work employs an output-feedback control design, which relies solely on measurable outputs rather than full-state information [26]. Motivated by the foregoing observations, this paper addresses the master-slave synchronization problem for uncertain fractional-order Lur'e systems (FOLSs) subject to time-varying input delay. To the best of our knowledge, this is the first study to simultaneously account for FO dynamics, input delay, output-feedback control, and Lur'e system synchronization within a unified framework– a combination that has not been previously reported in the literature.

The remainder of this paper is organized as follows. Section 2 provides the necessary preliminaries. Section 3 formulates the synchronization problem. Section 4 presents the derived conditions along with their proofs. Sections 5 and 6 present the numerical simulation results and the conclusion, respectively.

We adopt the following notation: the symbol $*$ marks the symmetric component of a matrix and $(\cdot)^T$ denotes transposition.

Remark 1: The Caputo derivative is a widely used approach for defining fractional derivatives [27]. A key advantage of Caputo's definition is that it allows initial conditions to be specified in the same form as for integer-order derivatives, thereby preserving their well-established physical interpretations. Furthermore, this definition is generally more straightforward to implement in practical applications and numerical simulations, since its formulation relies solely on standard calculus operations. In this work, we adopt the Caputo fractional derivative to model the fractional dynamics of the Lur'e systems.

II Preliminaries

Definition 1 [3]. The Caputo definition of the FO derivative for $x(t)$ is given by:

$$ {}^{C}\mathcal{D}^{\vartheta}x(t) = \frac{1}{\Gamma(1-\vartheta)}\int_0^t (t-\theta)^{-\vartheta}x'(\theta)d\theta \quad (1)$$

in which $0 < \vartheta < 1$ and $\Gamma(\theta) = \int_0^\infty t^{\theta-1}e^{-t}dt$ represents the Gamma function, and $x(t)$ is continuous throughout its domain.

Lemma 1 [28]: For a vector of differentiable function $X(t) \in \mathbb{R}^n$ the following relationship holds:

$$ {}^{C}\mathcal{D}^{\vartheta}(X^T(t)\psi X(t)) \le 2X^T(t)\,\psi({}^{C}\mathcal{D}^{\vartheta}X(t)) \quad (2)$$

where $\vartheta \in (0,1)$ and $\psi \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix.

Lemma 2 [29]: Let Θ_1, Θ_2, and Θ_3 be given matrices, where $\Theta_1 = {\Theta_1}^T$ and $\Theta_2 > 0$, then $\Theta_1 + {\Theta_3}^T{\Theta_2}^{-1}\Theta_3 < 0$ if and only if

$$\begin{pmatrix}\Theta_1 & {\Theta_3}^T\\ \Theta_3 & -\Theta_2\end{pmatrix} < 0 \text{ or } \begin{pmatrix}-\Theta_2 & \Theta_3\\ {\Theta_3}^T & \Theta_1\end{pmatrix} < 0 \quad (3)$$

Lemma 3 [30]: For given matrices Φ, $\mathcal{H}$, and $Q = Q^T$ with compatible sizes and all $G(t)$ fulfilling $G^T(t)G(t)$, one has

$$Q + \mathcal{H}^T G^T(t)\Phi^T + \Phi G(t)\mathcal{H} < 0 \quad (4)$$

if and only if a scalar $\gamma > 0$ exists in such a way that

$$Q + \gamma\Phi\Phi^T + \gamma^{-1}\mathcal{H}^T\mathcal{H} < 0 \quad (5)$$

Lemma 4 [31] (Razumikhin stability theorem): For the given system

$$ {}^{C}\mathcal{D}^{\vartheta}X(t) = f(t, X_t) \quad (6)$$

where $X_t = X(t+\theta)$, $-\wp \le \theta \le 0$. Suppose the nondecreasing continuous functions $\{\xi_1, \xi_2, \xi_3\} : \mathbb{R}^+ \to \mathbb{R}^+$, and also the functions $\xi_1(z)$ and $\xi_2(z)$ being positive for $z > 0$, with $\xi_1(0) = \xi_2(0) = 0$ and ξ_2 is strictly increasing. If there exists a constant $n > 1$ and a continuously differentiable function $V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$ such that

$$\begin{aligned}&\xi_1(\|X\|) \le V(t,X) \le \xi_2(\|X\|)\\ &{}^{C}\mathcal{D}^{\vartheta}V(t,X(t)) \le -\xi_3(\|X(t)\|)\\ &if \quad V(t+\delta, X(t+\delta)) \le nV(t,X(t)),\ \ t \ge 0,\ \forall\delta \in [-\wp, 0]\end{aligned} \quad (7)$$

the zero solution of system (6) is asymptotically stable.

III Problem formulation

Consider the following master-slave synchronous uncertain FO Lur'e system using non-fragile delayed static output-feedback control

$$\mathrm{M}: \begin{cases}{}^{C}\mathcal{D}^{\vartheta}X(t) = (\mathcal{A}+\Delta\mathcal{A})X(t)\\ \qquad\qquad +(\mathcal{B}+\Delta\mathcal{B})\phi(DX(t))\\ X(t) = \psi_m(t)\\ q(t) = CX(t)\end{cases} \quad (8)$$

$$\mathrm{S}: \begin{cases}{}^{C}\mathcal{D}^{\vartheta}Y(t) = (\mathcal{A}+\Delta\mathcal{A})Y(t)\\ \qquad\qquad +(\mathcal{B}+\Delta\mathcal{B})\phi(DY(t)) + u(t)\\ Y(t) = \psi_s(t)\\ p(t) = CY(t)\end{cases} \quad (9)$$

$$u(t) = (L+\Delta L)(q(t-\wp(t)) - p(t-\wp(t))) \quad (10)$$

where $0 < \vartheta < 1$, the master system denoted by M has state $X(t) \in \mathbb{R}^n$ and output $q(t) \in \mathbb{R}^m$, while the slave system denoted by S is defined by state $Y(t) \in \mathbb{R}^n$ and output $p(t) \in \mathbb{R}^m$. The system matrices $\mathcal{A} \in \mathbb{R}^{n\times n}$, $\mathcal{B} \in \mathbb{R}^{n\times n_h}$, $D \in \mathbb{R}^{n_h\times n}$ and $C \in \mathbb{R}^{m\times n}$ are given constant with suitable sizes. Additionally, the uncertainty terms are expressed by

$$[\Delta\mathcal{A} \quad \Delta\mathcal{B}] = \mathcal{H}G(t)[\mathcal{E}_0 \quad \mathcal{E}_1] \quad (11)$$

in which the known real matrices $\mathcal{H}, \mathcal{E}_0$, and $\mathcal{E}_1$ are of compatible dimensions and the time-varying parametric uncertainties are reflected onto $G(t)$, which satisfies $G^T(t)G(t) \le I$.

The non-fragile control law $u(t) \in \mathbb{R}^n$ employs a delayed output feedback strategy and is applied to the slave system. The controller gain is denoted by $L \in \mathbb{R}^{n\times l}$, with $\Delta L = \mathcal{H}_L G(t)\mathcal{E}_L$, where $\mathcal{H}_L$ and $\mathcal{E}_L$ are known constant matrices of appropriate dimensions, and $G(t)$ satisfies $G^T(t)G(t) \leq I$. The initial condition is specified as $\psi(t) \in C([-\wp, 0], \mathbb{R}^n)$, where $\wp$ denotes the bound of the time-varying input delay. The nonlinearity $\phi(\cdot) : \mathbb{R}^{n_h} \to \mathbb{R}^{n_h}$ is assumed to be a diagonal mapping, where each entry $\phi_j(\cdot)$ satisfies a sector constraint in the interval $[0, k_j]$. More precisely, for any $a, b \in \mathbb{R}$ and for each $j = 1, 2, \ldots, n_h$, the following inequality holds:

$$0 \leq \frac{\phi_j(a) - \phi_j(b)}{a - b} \leq {k_j}^+ \tag{12}$$

where $j = 1, 2, ..., \ell$. Let $e(t) = Y(t) - X(t)$ denote the master-slave synchronization error.

Accordingly, the synchronization error dynamics are described as:

$$\begin{aligned} {}^C\mathcal{D}^\vartheta e(t) = (\mathcal{A} + \Delta\mathcal{A})e(t) + (\mathcal{B} + \Delta\mathcal{B})g(De(t), X(t)) \\ - (L + \Delta L)Ce(t - \wp(t)) \end{aligned} \tag{13}$$

where the nonlinear function $g(De(t), X(t)) = \phi(DX(t) + De(t)) - \phi(DX(t))$ holds the interval $\left[0, k_j\right]$. Then, we have

$$0 \leq \frac{\phi_j(D_j(e(t) + X(t))) - \phi_j(D_j X(t))}{D_j e(t)} \leq {k_j}^+ \tag{14}$$

where $D_j e(t) \neq 0$, $j = 1, 2, \ldots, \ell$, and D_j represents the j-th row of D.

From (14), we get:

$$g_j(D_j e(t), X(t)) \left[g_j(D_j e(t), X(t)) - {k_j}^+ D_j e(t)\right] \leq 0 \tag{15}$$

The main objective of this paper is to design a controller (10) that achieves synchronization between the master system (8) and the slave system (9). In other words, we seek to determine the controller gain matrix L such that the error system (13) is asymptotically stable, ensuring that the master and slave systems synchronize.

IV Main results

In the following theorem, we propose an output-feedback control strategy to stabilize the error-state equation (13).

Theorem: Consider the error dynamics (13). For given prescribed matrices $K = \text{diag}\{k_1, \ldots, k_{n_h}\}$, if there exist a positive scalar γ, an integer $n > 1$, a symmetric positive definite matrix $\bar{P} \in \mathbb{R}^{n\times n}$, a positive diagonal matrix $\bar{T} = \text{diag}\{t_1, \ldots, t_{n_h}\} \in \mathbb{R}^{n\times n}$, and a matrix $\mathcal{X} \in \mathbb{R}^{n\times n}$ of appropriate dimensions, satisfying the following conditions:

$$\Upsilon = \begin{pmatrix} \Upsilon_{11} & \Upsilon_{12} & \Upsilon_{13} & \Upsilon_{14} & \Upsilon_{15} & \Upsilon_{16} & 0 & \Upsilon_{18} \\ * & \Upsilon_{22} & \Upsilon_{23} & \Upsilon_{24} & 0 & \Upsilon_{26} & \Upsilon_{27} & \Upsilon_{28} \\ * & * & \Upsilon_{33} & \Upsilon_{34} & \Upsilon_{35} & 0 & 0 & 0 \\ * & * & * & \Upsilon_{44} & 0 & \Upsilon_{46} & 0 & \Upsilon_{48} \\ * & * & * & * & \Upsilon_{55} & 0 & 0 & 0 \\ * & * & * & * & * & \Upsilon_{66} & 0 & 0 \\ * & * & * & * & * & * & \Upsilon_{77} & 0 \\ * & * & * & * & * & * & * & \Upsilon_{88} \end{pmatrix} < 0 \tag{16}$$

where

$$\begin{aligned}
&\Upsilon_{11} = \mathcal{A}\bar{P} + \bar{P}\mathcal{A}^T + n\bar{P},\ \Upsilon_{12} = -\mathcal{X} + \bar{P}\mathcal{A}^T \\
&\Upsilon_{13} = \mathcal{B}\bar{T} + \bar{P}D^T K,\ \Upsilon_{14} = \bar{P}\mathcal{A}^T \\
&\Upsilon_{15} = \bar{P}{\mathcal{E}_0}^T,\ \Upsilon_{16} = \gamma_1\mathcal{H} \\
&\Upsilon_{18} = -\gamma_2\mathcal{H}_L,\ \Upsilon_{22} = -\bar{P} - \mathcal{X} - \mathcal{X}^T \\
&\Upsilon_{23} = \mathcal{B}\bar{T},\ \Upsilon_{24} = -\mathcal{X}^T - \bar{P} \\
&\Upsilon_{26} = \gamma_1\mathcal{H},\ \Upsilon_{27} = \bar{P}C^T{\mathcal{E}_L}^T \\
&\Upsilon_{28} = -\gamma_2\mathcal{H}_L,\ \Upsilon_{33} = -2\bar{T} \\
&\Upsilon_{34} = \bar{T}\mathcal{B}^T,\ \Upsilon_{35} = \bar{T}{\mathcal{E}_1}^T, \\
&\Upsilon_{44} = -2\bar{P},\ \Upsilon_{46} = \gamma_1\mathcal{H} \\
&\Upsilon_{48} = -\gamma_2\mathcal{H}_L,\ \Upsilon_{55} = \Upsilon_{66} = -\gamma_1 I, \\
&\Upsilon_{77} = \Upsilon_{88} = -\gamma_2 I
\end{aligned}$$

The proposed output-feedback controller (10) is designed with gain matrices

$$L = \mathcal{X}\bar{P}^{-1}C^T\left(CC^T\right)^{-1} \tag{17}$$

Proof: Choose the following Lyapunov function candidate

$$V(t) = e^T(t)Pe(t) \tag{18}$$

Calculating the fractional derivative of (18) using Lemma 1 yields:

$$\begin{aligned} {}^C\mathcal{D}^\vartheta V(t) \leq\ & 2e^T(t)P(\mathcal{A} + \Delta\mathcal{A})e(t) \\ & + 2e^T(t)P(\mathcal{B} + \Delta\mathcal{B})g(De(t), X(t)) \\ & - 2e^T(t)P(L + \Delta L)Ce(t - \wp(t)) \end{aligned} \tag{19}$$

From (15), for any diagonal positive matrix,

$$\begin{aligned} 2\Big[&e^T(t)D^T KTg\left(De(t), X(t)\right) \\ & - g^T\left(De(t), X(t)\right)Tg\left(De(t), X(t)\right)\Big] \geq 0 \end{aligned} \tag{20}$$

holds. From (13), the following expression holds

$$\begin{aligned} & 2\left(e^T(t - \wp(t)) + {}^C\mathcal{D}^\vartheta e^T(t)\right)P \\ & \times \Big(- {}^C\mathcal{D}^\vartheta e(t) + (\mathcal{B} + \Delta\mathcal{B})g(\mathcal{D}e(t), X(t)) \\ & \quad + (\mathcal{A} + \Delta\mathcal{A})e(t) - (L + \Delta L)Ce(t - \wp(t))\Big) = 0 \end{aligned} \tag{21}$$

According to Lemma 4, whenever $e(t)$ satisfies

$$V(t + \theta, e(t + \theta)) < nV(t, e(t)),\ -\wp \leq \theta \leq 0 \tag{22}$$

Then, for some $n > 1$, one has

$$ne^T(t)Pe(t) - e^T(t - \wp(t))Pe(t - \wp(t)) \geq 0 \tag{23}$$

Integrating (19) with (20),(21) and (23) yields

$$\begin{aligned}
{}^{C}\mathcal{D}^{\vartheta}V(t) \le\ & 2e^{T}(t)P(\mathcal{A}+\Delta\mathcal{A})e(t) \\
& + 2e^{T}(t)P(\mathcal{B}+\Delta\mathcal{B})g(De(t),X(t)) \\
& - 2e^{T}(t)P(L+\Delta L)Ce(t-\wp(t)) \\
& + 2e^{T}(t)D^{T}KTg(De(t),X(t)) \\
& - 2g^{T}(De(t),X(t))Tg(De(t),X(t)) \\
& + ne^{T}(t)Pe(t) - e^{T}(t-\wp(t))Pe(t-\wp(t)) \\
& - 2e^{T}(t-\wp(t))P\,{}^{C}\mathcal{D}^{\vartheta}e(t) \\
& + 2e^{T}(t-\wp(t))P(\mathcal{A}+\Delta\mathcal{A})e(t) \\
& + 2e^{T}(t-\wp(t))P(\mathcal{B}+\Delta\mathcal{B})g(De(t),X(t)) \\
& - 2e^{T}(t-\wp(t))P(L+\Delta L)Ce(t-\wp(t)) \\
& - 2\,{}^{C}\mathcal{D}^{\vartheta}e^{T}(t)P\,{}^{C}\mathcal{D}^{\vartheta}e(t) \\
& + 2\,{}^{C}\mathcal{D}^{\vartheta}e^{T}(t)P(\mathcal{A}+\Delta\mathcal{A})e(t) \\
& + 2\,{}^{C}\mathcal{D}^{\vartheta}e^{T}(t)P(\mathcal{B}+\Delta\mathcal{B})g(De(t),X(t)) \\
& - 2\,{}^{C}\mathcal{D}^{\vartheta}e^{T}(t)P(L+\Delta L)Ce(t-\wp(t)) \\
\le\ & \Xi^{T}(t)\,\bar{\Upsilon}\Xi(t)
\end{aligned} \tag{24}$$

where

$$\Xi^{T}(t) = \begin{bmatrix} e^{T}(t) & e^{T}(t-\wp(t)) & g^{T}(De(t),X(t)) & {}^{C}\mathcal{D}^{\vartheta}e^{T}(t) \end{bmatrix} \tag{25}$$

$$\bar{\Upsilon} = \begin{pmatrix} \bar{\Upsilon}_{11} & \bar{\Upsilon}_{12} & \bar{\Upsilon}_{13} & \bar{\Upsilon}_{14} \\ * & \bar{\Upsilon}_{22} & \bar{\Upsilon}_{23} & \bar{\Upsilon}_{24} \\ * & * & \bar{\Upsilon}_{33} & \bar{\Upsilon}_{34} \\ * & * & * & \bar{\Upsilon}_{44} \end{pmatrix} < 0 \tag{26}$$

with

$$\begin{aligned}
\bar{\Upsilon}_{11} &= P\mathcal{A} + P\Delta\mathcal{A} + \mathcal{A}^{T}P + \Delta\mathcal{A}^{T}P + nP \\
\bar{\Upsilon}_{12} &= -PLC - P\Delta LC + \mathcal{A}^{T}P + \Delta\mathcal{A}^{T}P \\
\bar{\Upsilon}_{13} &= P\mathcal{B} + P\Delta\mathcal{B} + D^{T}KT,\ \bar{\Upsilon}_{14} = \mathcal{A}^{T}P + \Delta\mathcal{A}^{T}P \\
\bar{\Upsilon}_{22} &= -P - PLC - P\Delta LC - C^{T}L^{T}P - C^{T}\Delta L^{T}P \\
\bar{\Upsilon}_{23} &= P\mathcal{B} + P\Delta\mathcal{B},\ \bar{\Upsilon}_{24} = -C^{T}L^{T}P - C^{T}\Delta L^{T}P - P \\
\bar{\Upsilon}_{33} &= -2T,\ \bar{\Upsilon}_{34} = \mathcal{B}^{T}P + \Delta\mathcal{B}^{T}P,\ \bar{\Upsilon}_{44} = -2P
\end{aligned}$$

In fact, (26) is equivalent to the following expression

$$\begin{aligned}
&\begin{pmatrix} \hat{\Upsilon}_{11} & \hat{\Upsilon}_{12} & \hat{\Upsilon}_{13} & \hat{\Upsilon}_{14} \\ * & \hat{\Upsilon}_{22} & \hat{\Upsilon}_{23} & \hat{\Upsilon}_{24} \\ * & * & \hat{\Upsilon}_{33} & \hat{\Upsilon}_{34} \\ * & * & * & \hat{\Upsilon}_{44} \end{pmatrix} \\
&+ \begin{pmatrix} P\mathcal{H} \\ P\mathcal{H} \\ 0 \\ P\mathcal{H} \end{pmatrix} G(t) \begin{pmatrix} \mathcal{E}_0 & 0 & \mathcal{E}_1 & 0 \end{pmatrix} \\
&+ \begin{pmatrix} \mathcal{E}_0^{T} \\ 0 \\ \mathcal{E}_1^{T} \\ 0 \end{pmatrix} G^{T}(t) \begin{pmatrix} \mathcal{H}^{T}P & \mathcal{H}^{T}P & 0 & \mathcal{H}^{T}P \end{pmatrix} \\
&+ \begin{pmatrix} -P\mathcal{H}_L \\ -P\mathcal{H}_L \\ 0 \\ -P\mathcal{H}_L \end{pmatrix} G(t) \begin{pmatrix} 0 & \mathcal{E}_L C & 0 & 0 \end{pmatrix} \\
&+ \begin{pmatrix} 0 \\ C^{T}\mathcal{E}_L^{T} \\ 0 \\ 0 \end{pmatrix} G^{T}(t) \begin{pmatrix} -\mathcal{H}_L^{T}P & -\mathcal{H}_L^{T}P & 0 & -\mathcal{H}_L^{T}P \end{pmatrix},
\end{aligned} \tag{27}$$

in which

$\hat{\Upsilon}_{11} = P\mathcal{A} + \mathcal{A}^{T}P + nP$, $\hat{\Upsilon}_{12} = -PLC + \mathcal{A}^{T}P$, $\hat{\Upsilon}_{13} = P\mathcal{B} + D^{T}KT$, $\hat{\Upsilon}_{14} = \mathcal{A}^{T}P$, $\hat{\Upsilon}_{22} = -P - PLC - C^{T}L^{T}P$, $\hat{\Upsilon}_{23} = P\mathcal{B}$, $\hat{\Upsilon}_{24} = -C^{T}L^{T}P - P$, $\hat{\Upsilon}_{33} = -2T$, $\hat{\Upsilon}_{34} = \mathcal{B}^{T}P$, $\hat{\Upsilon}_{44} = -2P$.

Based upon Lemma 3, we can achieve positive numbers γ_i, $i = 1, 2$, can be found satisfying

$$\begin{aligned}
&\begin{pmatrix} \hat{\Upsilon}_{11} & \hat{\Upsilon}_{12} & \hat{\Upsilon}_{13} & \hat{\Upsilon}_{14} \\ * & \hat{\Upsilon}_{22} & \hat{\Upsilon}_{23} & \hat{\Upsilon}_{24} \\ * & * & \hat{\Upsilon}_{33} & \hat{\Upsilon}_{34} \\ * & * & * & \hat{\Upsilon}_{44} \end{pmatrix} \\
&+ \gamma_1 \begin{pmatrix} P\mathcal{H} \\ P\mathcal{H} \\ 0 \\ P\mathcal{H} \end{pmatrix} \begin{pmatrix} \mathcal{H}^{T}P & \mathcal{H}^{T}P & 0 & \mathcal{H}^{T}P \end{pmatrix} \\
&+ \gamma_1{}^{-1} \begin{pmatrix} \mathcal{E}_0^{T} \\ 0 \\ \mathcal{E}_1^{T} \\ 0 \end{pmatrix} \begin{pmatrix} \mathcal{E}_0 & 0 & \mathcal{E}_1 & 0 \end{pmatrix} \\
&+ \gamma_2 \begin{pmatrix} -P\mathcal{H}_L \\ -P\mathcal{H}_L \\ 0 \\ -P\mathcal{H}_L \end{pmatrix} \begin{pmatrix} -\mathcal{H}_L^{T}P & -\mathcal{H}_L^{T}P & 0 & -\mathcal{H}_L^{T}P \end{pmatrix} \\
&+ \gamma_2{}^{-1} \begin{pmatrix} 0 \\ C^{T}\mathcal{E}_L^{T} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & \mathcal{E}_L C & 0 & 0 \end{pmatrix} \\
&< 0.
\end{aligned} \tag{28}$$

Using Lemma 2 (28) equivalently writes as

$$\begin{pmatrix}
\tilde{\Upsilon}_{11} & \tilde{\Upsilon}_{12} & \tilde{\Upsilon}_{13} & \tilde{\Upsilon}_{14} & \tilde{\Upsilon}_{15} & \tilde{\Upsilon}_{16} & 0 & \tilde{\Upsilon}_{18} \\
* & \tilde{\Upsilon}_{22} & \tilde{\Upsilon}_{23} & \tilde{\Upsilon}_{24} & 0 & \tilde{\Upsilon}_{26} & 0 & \tilde{\Upsilon}_{28} \\
* & * & \tilde{\Upsilon}_{33} & \tilde{\Upsilon}_{34} & \tilde{\Upsilon}_{35} & 0 & 0 & 0 \\
* & * & * & \tilde{\Upsilon}_{44} & 0 & \tilde{\Upsilon}_{46} & 0 & \tilde{\Upsilon}_{48} \\
* & * & * & * & \tilde{\Upsilon}_{55} & 0 & 0 & 0 \\
* & * & * & * & * & \tilde{\Upsilon}_{66} & 0 & 0 \\
* & * & * & * & * & * & \tilde{\Upsilon}_{77} & 0 \\
* & * & * & * & * & * & * & \tilde{\Upsilon}_{88}
\end{pmatrix} \tag{29}$$

where

$$\tilde{\Upsilon}_{11} = P\mathcal{A} + \mathcal{A}^T P + nP,\ \tilde{\Upsilon}_{12} = -PLC + \mathcal{A}^T P$$
$$\tilde{\Upsilon}_{13} = P\mathcal{B} + D^T KT,\ \tilde{\Upsilon}_{14} = \mathcal{A}^T P,\ \tilde{\Upsilon}_{15} = \mathcal{E}_0{}^T$$
$$\tilde{\Upsilon}_{16} = \gamma_1 P\mathcal{H},\ \tilde{\Upsilon}_{18} = -\gamma_2 P\mathcal{H}_L,$$
$$\tilde{\Upsilon}_{22} = -P - PLC - C^T L^T P,\ \tilde{\Upsilon}_{23} = P\mathcal{B}$$
$$\tilde{\Upsilon}_{24} = -C^T L^T P - P,\ \tilde{\Upsilon}_{26} = \gamma_1 P\mathcal{H}$$
$$\tilde{\Upsilon}_{27} = C^T \mathcal{E}_L{}^T,\ \tilde{\Upsilon}_{28} = -\gamma_2 P\mathcal{H}_L$$
$$\tilde{\Upsilon}_{33} = -2T,\ \tilde{\Upsilon}_{34} = \mathcal{B}^T P,\ \tilde{\Upsilon}_{35} = \mathcal{E}_1{}^T$$
$$\tilde{\Upsilon}_{44} = -2P,\ \tilde{\Upsilon}_{46} = \gamma_1 P\mathcal{H}$$
$$\tilde{\Upsilon}_{48} = -\gamma_2 P\mathcal{H}_L,\ \tilde{\Upsilon}_{55} = \tilde{\Upsilon}_{66} = -\gamma_1 I$$
$$\tilde{\Upsilon}_{77} = \tilde{\Upsilon}_{88} = -\gamma_2 I$$

Multiplying (29) from the left and right by $\mathrm{diag}\{P^{-1}, P^{-1}, T^{-1}, P^{-1}, I, I, I, I\}$ produces

$$\begin{pmatrix} \check{\Upsilon}_{11} & \check{\Upsilon}_{12} & \check{\Upsilon}_{13} & \check{\Upsilon}_{14} & \check{\Upsilon}_{15} & \check{\Upsilon}_{16} & 0 & \check{\Upsilon}_{18} \\ * & \check{\Upsilon}_{22} & \check{\Upsilon}_{23} & \check{\Upsilon}_{24} & 0 & \check{\Upsilon}_{26} & 0 & \check{\Upsilon}_{28} \\ * & * & \check{\Upsilon}_{33} & \check{\Upsilon}_{34} & \check{\Upsilon}_{35} & 0 & 0 & 0 \\ * & * & * & \check{\Upsilon}_{44} & 0 & \check{\Upsilon}_{46} & 0 & \check{\Upsilon}_{48} \\ * & * & * & * & \check{\Upsilon}_{55} & 0 & 0 & 0 \\ * & * & * & * & * & \check{\Upsilon}_{66} & 0 & 0 \\ * & * & * & * & * & * & \check{\Upsilon}_{77} & 0 \\ * & * & * & * & * & * & * & \check{\Upsilon}_{88} \end{pmatrix} \quad (30)$$

where

$$\check{\Upsilon}_{11} = \mathcal{A}P^{-1} + P^{-1}\mathcal{A}^T + nP^{-1}$$
$$\check{\Upsilon}_{12} = -LCP^{-1} + P^{-1}\mathcal{A}^T$$
$$\check{\Upsilon}_{13} = \mathcal{B}T^{-1} + P^{-1}D^T K$$
$$\check{\Upsilon}_{14} = P^{-1}\mathcal{A}^T,\ \check{\Upsilon}_{15} = P^{-1}\mathcal{E}_0{}^T$$
$$\check{\Upsilon}_{16} = \gamma_1\mathcal{H},\ \check{\Upsilon}_{18} = -\gamma_2\mathcal{H}_L$$
$$\check{\Upsilon}_{22} = -P^{-1} - LCP^{-1} - P^{-1}C^T L^T$$
$$\check{\Upsilon}_{23} = \mathcal{B}T^{-1},\ \check{\Upsilon}_{24} = -P^{-1}C^T L^T - P^{-1}$$
$$\check{\Upsilon}_{26} = \gamma_1\mathcal{H},\ \check{\Upsilon}_{27} = P^{-1}C^T\mathcal{E}_L{}^T,\ \check{\Upsilon}_{28} = -\gamma_2\mathcal{H}_L$$
$$\check{\Upsilon}_{33} = -2T^{-1},\ \check{\Upsilon}_{34} = T^{-1}\mathcal{B}^T,\ \check{\Upsilon}_{35} = T^{-1}\mathcal{E}_1{}^T$$
$$\check{\Upsilon}_{44} = -2P^{-1},\ \check{\Upsilon}_{46} = \gamma_1\mathcal{H},\ \check{\Upsilon}_{48} = -\gamma_2\mathcal{H}_L$$
$$\check{\Upsilon}_{55} = \check{\Upsilon}_{66} = -\gamma_1 I$$
$$\check{\Upsilon}_{77} = \check{\Upsilon}_{88} = -\gamma_2 I$$

Considering $n > 1, \bar{P} = P^{-1}, \bar{T} = T^{-1}$ and $X = LCP^{-1}$ we obtain the condition (16). Applying Lemma 4 ensures asymptotic stability of the error system (13).

V Simulations

The validity and feasibility of the proposed approach are illustrated through a numerical example. MATLAB is used for the simulations, where the modified Adams–Bashforth method [32] is employed to solve the fractional-order differential equations.

Example: Consider the chaotic neural networks in the form of Lur'e system with following matrices

$$\mathcal{A} = \begin{bmatrix} -2.5 & 5 & 0 \\ 1 & -1 & 1 \\ 0 & -4 & 0 \end{bmatrix},\ \mathcal{B} = \begin{bmatrix} -4 \\ 0 \\ 0 \end{bmatrix}$$

$$D^T = C^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ \mathcal{E}_0 = \begin{pmatrix} 0.2 & 0.4 & 0.2 \end{pmatrix}$$

$$\mathcal{H} = \begin{pmatrix} -0.1 \\ -0.2 \\ 0.1 \end{pmatrix},\ \mathcal{E}_1 = -0.3,\ G(t) = \cos(t)$$

$$\mathcal{H}_L = \begin{pmatrix} -0.1 \\ 0.2 \\ 0.1 \end{pmatrix}, \mathcal{E}_L = 0.3$$

Moreover, neuron activation functions are given by the nonlinearity $\phi(X_1(t)) = \frac{1}{2}(|X_1 + 1| - |X_1 - 1|)$ lies in the sector $[0, 1]$. By selecting $\wp(t) = 0.2 + 0.4|\sin(t)|$ and solving the condition in Theorem, we derive the following results for $\vartheta = 0.9$:

$$\bar{P} = \begin{pmatrix} 0.9644 & -0.1669 & -0.2320 \\ -0.1669 & 0.1754 & 0.0502 \\ -0.2320 & 0.0502 & 0.3760 \end{pmatrix}$$
$$X = \begin{pmatrix} 2.8432 & -0.6336 & -0.4794 \\ -0.7654 & 0.5326 & -0.0486 \\ -0.3622 & 0.2024 & 0.2833, \end{pmatrix}$$
$$\bar{T} = 0.0412$$

The proposed design procedure yields the output-feedback controller gain matrice

$$L = \begin{bmatrix} 2.9342 & -0.5071 & -0.0771 \end{bmatrix}^T$$

The initial conditions for the master and slave systems are chosen as $X(0) = \begin{bmatrix} -0.4 & 0.5 & -0.5 \end{bmatrix}^T$ and $Y(0) = \begin{bmatrix} 0.2 & -0.3 & 0.6 \end{bmatrix}^T$, respectively. The results are presented in Figs. 1 and 2. Figure 1 depicts the state trajectories of both the master and slave systems, clearly showing that the slave states converge to the master states over time, thereby confirming the effectiveness of the proposed synchronization scheme. The corresponding synchronization error is shown in Fig. 2, which clearly illustrates the asymptotic convergence of the error to zero. These results confirm that master-slave synchronization is successfully achieved.

VI Conclusion

This paper has addressed the synchronization problem for uncertain FOLSs with time-varying input delay using a non-fragile output-feedback controller. Sufficient conditions, expressed in terms of LMIs, are derived using a Lyapunov-based approach. In particular, the proposed scheme ensures asymptotic synchronization while accommodating both system uncertainties and controller gain variations. Future work will focus on extending the method to more general FO systems, such as those with variable fractional orders; adapting the control framework to synchronization problems in larger-scale networks, including multi-agent systems and interconnected dynamical networks; and considering more general delay structures, such as distributed, stochastic, or neutral delays, which are common in practical applications. Numerical simulations confirm the effectiveness and robustness of the proposed method.

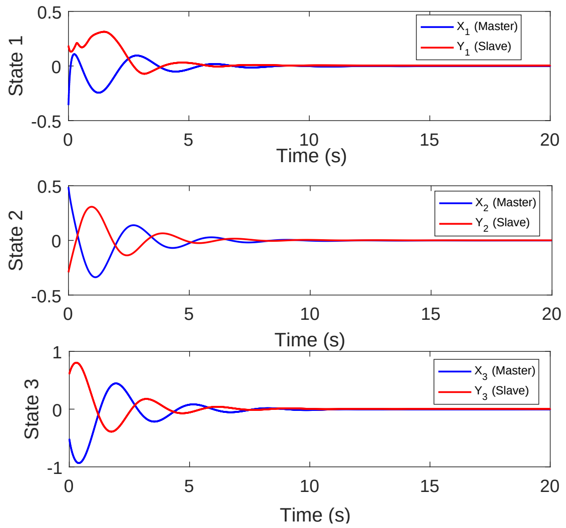

Figure 1. Master and slave state trajectories.

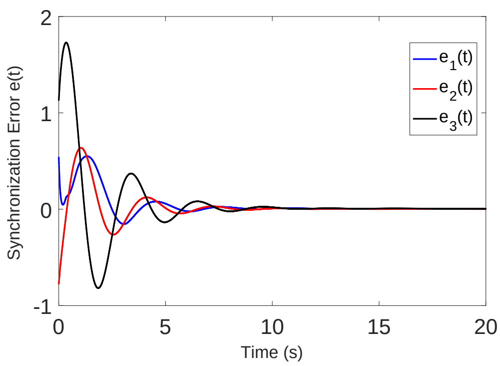

Figure 2. Synchronization errors.

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