Delayed PD Feedback Guaranteed Cost Control For Master–Slave Synchronization Of Variable Fractional-Order Lur'e Systems

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Abstract: **This paper proposes a guaranteed cost proportional-derivative (PD) feedback strategy for synchronizing master–slave variable fractional-order Lur'e systems (VFOLSs) when the systems are subject to input time delays. The design ensures both asymptotic convergence and a guaranteed upper bound on the cost performance. Based on linear matrix inequalities (LMIs) and the Lyapunov–Krasovskii stability approach, sufficient criteria for synchronization are presented. The effectiveness of the proposed control approach is demonstrated via simulation results, which also confirm its practical applicability.**

Keywords: **Variable fractional-order systems, Chaotic dynamics, Synchronization, Guaranteed cost, PD controller**

I Introduction

The issues of synchronization and control of chaotic systems have attracted considerable research interest owing to their broad applicability in chemical processes, secure communication systems and information technology [1]–[5]. In the context of secure communication, identical chaotic systems are adopted at both the transmitter (master) and receiver (slave) ends. The original signal is first encrypted by adding it to the system's chaotic output at the transmitter. Subsequently, the signal to be transmitted is retrieved at the receiver by subtracting the chaotic output, thereby recovering the original signal through decryption. However, for this process to yield successful and secure communication, the chaotic subsystems at the transmission and reception ends are required to be in complete synchronization. To date, several methods for synchronization have been established, including complete [6], lag [7], projective [8], and projective lag synchronization [9]. The Lur'e dynamical systems are a major class of nonlinear systems, defined by the combination of a linear dynamics in the forward branch and a sector-bounded nonlinearity in the feedback branch [10]. Consequently, the problems of Lur'e system chaos control and synchronization have received substantial research attention in recent years [11]–[13]. Meanwhile, time delay is an inherent phenomenon in real-world implementations and is known to render the dynamical behavior of systems considerably more complex [14, 15]. Recognizing this, the authors of [16] investigated the influence of time delays on the synchronization of Lur'e chaotic systems. Since then, time-delayed chaos synchronization in Lur'e systems has been extensively reported by numerous researchers.

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Fractional-order (FO) calculus has emerged as a prominent research topic over the past several decades, experiencing rapid theoretical and practical advancements [17]–[22]. This sustained interest stems largely from its distinct advantages over classical integer-order calculus, particularly in reflecting the memory behavior and inherited characteristics in dynamical systems. As a recent development, variable fractional-order (VFO) calculus has emerged as a generalization of the constant-order fractional framework and has gained considerable attention across a diverse range of disciplines, including materials science, chemistry, and electrochemistry, among others [23]–[25]. For instance, variable-order differential equations have been employed to characterize the history-dependent drag expression in certain physical processes [26, 27]. A key advantage of VFO calculus is having an adaptive memory for describing nonlinear phenomena, which is a valuable tool for simulating spatial or temporal correlation phenomena [28].

FO Lur'e system synchronization has attracted considerable interest in the literature [29, 30]. Notably, the previous findings are not generally valid for VFO systems. The master–slave synchronization of variable fractional-order Lur'e systems (VFOLSs) has also received little attention so far, which forms the central motivation for this paper. A wide variety of theoretical and experimental methods have been established for synchronizing chaotic systems, including linear and active feedback [31], adaptive [32] and impulsive [33] control, backstepping [34], and sampled-data control [35]. The guaranteed cost control concept, pioneered by Chang and Peng [36], provides a framework for constructing a controller that ensures robust stabilization of systems subject to uncertainty, while ensuring a prescribed level of performance. This methodology has recently been extended to integer- and FO systems [37]–[40]. Considerable research interest has been devoted to chaotic phenomena in FO systems, especially regarding synchronization problems. Several methods have been proposed, including an adaptive scheme [41], sliding mode control [42], and a linear feedback control [43]. Owing to its conceptual simplicity and well-established parameter tuning rules, the proportional-integral-derivative (PID) controllers and their extended forms have become common tools in addressing master-slave synchronization problems [44, 45]. In particular, the proportional-derivative (PD) controller offers the advantages of fast transient response and improved system stability, making it a particularly attractive choice for the problem under consideration [46]–[48]. Finally, to the best of our knowledge, this is the first work that simultaneously addresses VFO, input delay, guaranteed cost PD control, and synchronization of Lur'e systems in a unified framework, a combination of features not reported in the existing literature [49, 50]. Here, the symbol $*$ marks the symmetric component of a matrix, $\lambda_{\max}(X)$ gives the maximum real eigenvalue part, and $(\cdot)^T$ denotes transposition. The notation ${}^C\mathcal{D}^\nu$ is used throughout this paper as a shorthand for the variable-order Caputo derivative ${}^C\mathcal{D}^{\nu(t)}$.

II Mathematical Foundations

Definition 1 [51]: For $0 < \nu < 1$, The Caputo definition of the VFO derivative for $\varkappa(t)$ is given by:

$$ {}^C\mathcal{D}^\nu \varkappa(t) = \frac{1}{\Gamma(1-\nu)} \int_0^t (t-\theta)^{-\nu} \varkappa'(\theta) d\theta $$

in which $\Gamma(\theta) = \int_0^\infty t^{\theta-1} e^{-t} dt$ represents the Gamma function, and $\varkappa(t)$ is continuous throughout its domain.

Definition 2 [52]: The concept of guaranteed cost control is defined as follows. If a positive scalar J^* and a stabilizing control input $u^*(t)$ can be found such that $J < J^*$, then $u^*(t)$ is termed the guaranteed cost controller and J^* is the corresponding guaranteed cost bound.

Lemma 1 [53]: Define $\mathcal{Z}(t) \in \mathbb{R}^n$ as a differentiable vector-valued function with $\nu \in (0, 1)$, The following relationship holds:

$$ {}^C\mathcal{D}^\nu (\mathcal{Z}^T(t) \Psi \mathcal{Z}(t)) \le 2\mathcal{Z}^T(t)\, \Psi ({}^C\mathcal{D}^\nu \mathcal{Z}(t)) $$

where Ψ is an $n \times n$ symmetric positive definite matrix.

Lemma 2 [54]: Let Θ_1, Θ_2, and Θ_3 be given matrices, where $\Theta_1 = {\Theta_1}^T$ and $\Theta_2 > 0$, then $\Theta_1 + {\Theta_3}^T {\Theta_2}^{-1} \Theta_3 < 0$ if and only if

$$ \begin{pmatrix} \Theta_1 & {\Theta_3}^T \\ \Theta_3 & -\Theta_2 \end{pmatrix} < 0 \text{ or } \begin{pmatrix} -\Theta_2 & \Theta_3 \\ {\Theta_3}^T & \Theta_1 \end{pmatrix} < 0 $$

Lemma 3 [55]: The following inequality, which will be used in the stability analysis, holds for any vectors a and δ in $\mathbb{R}^n$, and for any positive definite matrix Ω :

$$ \pm 2a^T \delta \le a^T \Omega a + \delta^T \Omega^{-1} \delta $$

Lemma 4 [56]: For the time delay VFO system

$$ {}^C\mathcal{D}^\nu \mathcal{X}(t) = f(t, \mathcal{X}_t) $$

with $\mathcal{X}_t(\sigma) = \mathcal{X}(t+\sigma)$ for $-\mu_m \le \sigma \le 0$, $0 < \nu_{min} \le \nu \le \nu_{max} < 1$, the equilibrium piont $\mathcal{X} = 0$ is asymptotically stable if there can be found a Lyapunov function $V(t)$ and class-$\mathcal{K}$ functions ϑ_1, ϑ_2, and ϑ_3 for which:

$$ \vartheta_1(\|\mathcal{X}(t)\|) \le V(t) \le \vartheta_2(\|\mathcal{X}(t)\|) $$

$$ {}^C\mathcal{D}^\nu V(t) \le -\vartheta_3(\|\mathcal{X}(t)\|), \quad 0 \le t $$

III Statement of the Problem

We introduce the following master and slave systems, respectively, regarding the synchronization of VFOLSs.

$$ \text{Master}: \begin{cases} {}^C\mathcal{D}^\nu \eta(t) = \mathcal{A}\eta(t) + \mathcal{H}\mathcal{F}(D\eta(t)) \\ \eta(t) = \psi_m(t), \quad t \in [-\mu, 0] \\ q(t) = C\eta(t) \end{cases} \tag{1} $$

$$ \text{Slave}: \begin{cases} {}^C\mathcal{D}^\nu Y(t) = \mathcal{A}Y(t) + \mathcal{H}\mathcal{F}(DY(t)) + u(t) \\ Y(t) = \psi_s(t), \quad t \in [-\mu, 0] \\ p(t) = C\,Y(t) \end{cases} \tag{2} $$

with a delayed feedback PD control

$$ \begin{aligned} u(t) = {} & K_2({}^C\mathcal{D}^\nu q(t-\mu) - {}^C\mathcal{D}^\nu p(t-\mu)) \\ & + K_1(q(t-\mu) - p(t-\mu)) \end{aligned} \tag{3} $$

where $0 < \nu_{\min} \le \nu \le \nu_{\max} < 1$, the master has state $\eta(t) \in \mathbb{R}^n$ and output $q(t) \in \mathbb{R}^m$, while the slave is defined by state $Y(t) \in \mathbb{R}^n$ and output $p(t) \in \mathbb{R}^m$. The system matrices $\mathcal{A} \in \mathbb{R}^{n \times n}, \mathcal{H} \in \mathbb{R}^{n \times n_h}, D \in \mathbb{R}^{n_h \times n}$ and $C \in \mathbb{R}^{m \times n}$ are given constant matrices with suitable sizes. The control law $u(t) \in \mathbb{R}^n$ is applied to the slave system, and the controller gains are represented by $K_1, K_2 \in \mathbb{R}^{n \times l}$. The initial function is specified as $\psi(t) \in C([-\mu, 0], \mathbb{R}^n)$, where $\mu > 0$ denotes the delay. The nonlinearity $\mathcal{F}(\cdot) : \mathbb{R}^{n_h} \to \mathbb{R}^{n_h}$ is assumed to be a diagonal mapping, where each entry $\mathcal{F}_j(\cdot)$ satisfies a sector constraint in the interval $[0, k_j]$. More precisely, for any a, b and for each $j = 1, 2, \ldots, n_h$, we have:

$$ 0 \le \frac{\mathcal{F}_j(a) - \mathcal{F}_j(b)}{a - b} \le {k_j}^+ \tag{4} $$

where $j = 1, 2, ..., \ell$. Let $e(t) = Y(t) - \eta(t)$ denote the master-slave synchronization error. Accordingly, the synchronization error dynamics are described as:

$$ \begin{aligned} {}^C\mathcal{D}^\nu e(t) = {} & \mathcal{A}e(t) + \mathcal{H}g(De(t), \eta(t)) \\ & - K_1 Ce(t-\mu) - K_2 C\, {}^C\mathcal{D}^\nu e(t-\mu) \end{aligned} \tag{5} $$

where the nonlinear function $g(De(t), \eta(t)) = \mathcal{F}(D\eta(t) + De(t)) - \mathcal{F}(D\eta(t))$ holds the interval $[0, k_j]$. Then, we have

$$ 0 \le \frac{\mathcal{F}_j(D_j(e(t) + \eta(t))) - \mathcal{F}_j(D_j\eta(t))}{D_j e(t)} \le {k_j}^+ \tag{6} $$

where $D_j e(t) \ne 0$, $j = 1, 2, \ldots, \ell$, and D_j represents the j-th row of D. From (6), we directly get:

$$ g_j(D_j e(t), \eta(t)) \left[g_j(D_j e(t), \eta(t)) - {k_j}^+ D_j e(t) \right] \le 0 \tag{7} $$

We choose the following cost function to be minimized:

$$J = \frac{1}{\Gamma(\nu)} \int_0^{\mathfrak{h}} (\mathfrak{h} - s)^{\nu - 1} (e^T(s) Q_1 e(s) + u^T(s) Q_2 u(s)) ds \tag{8}$$

where $\mathfrak{h} > \mu > 0$ is a given parameter, and Q_1, Q_2 are positive definite symmetric matrices of appropriate dimensions. The objective is to develop a PD controller of the form (3) that enforces asymptotic tracking of the master trajectory by the slave system, thus ensuring master-slave synchronization for the FOLSs in (1) and (2). The task reduces to finding gain matrices K_1 and K_2 thus the system becomes asymptotically stable, meaning the error (5) tends to zero over time and the cost function (8) satisfies the requirement of Definition 2.

IV Stability criteria

In what follows, we propose a guaranteed cost control strategy to stabilize the error state equation (5).

Theorem: Let us consider the error dynamics (5) associated with the cost matrices that are symmetric and positive definite $Q_1 \in \mathbb{R}^{n \times n}$ and $Q_2 \in \mathbb{R}^{m \times m}$ as in (8).For given scalars μ, $\nu_1 = \nu_{\min}$ and $\nu_2 = \min\{\nu_{\min}, (1 - \nu_{\max})\}$, prescribed matrices $K = \mathrm{diag}\{k_1, \ldots, k_{n_h}\}$, positive scalars $\epsilon_1, \epsilon_2, \epsilon_3$, and a positive constant λ, we assume the existence of a matrice $\bar{P} \in \mathbb{R}^{n \times n}$ that is symmetric and positive definite , a positive diagonal matrice $\bar{T} = \mathrm{diag}\{t_1, \ldots, t_{n_h}\} \in \mathbb{R}^{n \times n}$, and matrices $X \in \mathbb{R}^{n \times n}$ and $Z \in \mathbb{R}^{n \times n}$ of suitable sizes satisfying

$$\begin{pmatrix}
\Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} & \Gamma_{15} & \bar{P}Q_1 & 0 & 0 & 0 & 0 & \mu^{\nu_2}{\nu_1}^{-1}\bar{P}\mathcal{A}^T & \mu^{\nu_2}{\nu_1}^{-1}\bar{P} \\
* & \Gamma_{22} & \epsilon_1 \mathcal{H}\bar{T} & \Gamma_{24} & \Gamma_{25} & 0 & X^T & X^T & 0 & 0 & -\mu^{\nu_2}{\nu_1}^{-1}X^T & 0 \\
* & * & -2\bar{T} & \Gamma_{34} & \bar{T}\mathcal{H}^T\epsilon_2 & 0 & 0 & 0 & 0 & 0 & \mu^{\nu_2}{\nu_1}^{-1}\bar{T}\mathcal{H}^T & 0 \\
* & * & * & \Gamma_{44} & \Gamma_{45} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & \Gamma_{55} & 0 & 0 & 0 & Z^T{Q_2}^T & Z^T & -\mu^{\nu_2}{\nu_1}^{-1}Z^T & 0 \\
* & * & * & * & * & -Q_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -{Q_2}^{-1} & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & -\lambda^{-1}I & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & -\lambda I & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & -{Q_2}^{-1} & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & -\mu^{\nu_2}{\nu_1}^{-1}I & 0 \\
* & * & * & * & * & * & * & * & * & * & * & -\mu^{\nu_2}{\nu_1}^{-1}I
\end{pmatrix} < 0 \tag{9}$$

where $\Gamma_{11} = \mathcal{A}\bar{P} + \bar{P}\mathcal{A}^T$, $\Gamma_{12} = -X + \bar{P}\mathcal{A}^T\epsilon_1$, $\Gamma_{13} = \mathcal{H}\bar{T} + \bar{P}D^T K$, $\Gamma_{14} = \bar{P}\mathcal{A}^T\epsilon_3$, $\Gamma_{15} = -Z + \bar{P}\mathcal{A}^T\epsilon_2$, $\Gamma_{22} = -\epsilon_1 X - X^T\epsilon_1$, $\Gamma_{24} = -\epsilon_1\bar{P} - X^T\epsilon_3$, $\Gamma_{25} = -\epsilon_1 Z - X^T\epsilon_2$, $\Gamma_{34} = \bar{T}\mathcal{H}^T\epsilon_3$, $\Gamma_{44} = -2\bar{P}\epsilon_3$, $\Gamma_{45} = -\bar{P}\epsilon_2 - \epsilon_3 Z$, and $\Gamma_{55} = -\epsilon_2 Z - Z^T\epsilon_2$. The proposed guaranteed cost controller (3) is designed with gain matrices

$$K_1 = X\bar{P}^{-1}C^T\left(CC^T\right)^{-1}, \quad K_2 = Z\bar{P}^{-1}C^T\left(CC^T\right)^{-1} \tag{10}$$

and the resulting guaranteed cost bound is computed as

$$J^* = \lambda_{\max}(\bar{P}^{-1})\|\psi\|^2 \tag{11}$$

Proof: Consider the Lyapunov function

$$V(t) = e^T(t)Pe(t) \tag{12}$$

Taking the fractional derivative of (12) using Lemma 1 yields:

$$\begin{aligned}
&{}^C\mathcal{D}^\nu V(t) + e^T(t)\,Q_1 e\,(t) \\
&\quad + \left(K_1 Ce\,(t-\mu) + K_2 C\,{}^C\mathcal{D}^\nu e\,(t-\mu)\right)^T Q_2 \\
&\quad \times Q_2(K_1 Ce\,(t-\mu) + K_2 C\,{}^C\mathcal{D}^\nu e\,(t-\mu)) \\
&\le -2e^T(t)PK_1Ce\,(t-\mu) \\
&\quad + 2e^T(t)P\mathcal{A}e\,(t) \\
&\quad + 2e^T(t)P\mathcal{H}g(De\,(t), \eta(t)) \\
&\quad - 2e^T(t)PK_2C^C\mathcal{D}^\nu e\,(t-\mu) \\
&\quad + e^T(t)\,Q_1 e\,(t) \\
&\quad + e^T(t-\mu)(K_1C)^T Q_2(K_1C)\,e\,(t-\mu) \\
&\quad + e^T(t-\mu)(K_1C)^T Q_2(K_2C)\,{}^C\mathcal{D}^\nu e\,(t-\mu) \\
&\quad + {}^C\mathcal{D}^\nu e^T(t-\mu)\,C^T {K_2}^T Q_2 K_1 Ce\,(t-\mu) \\
&\quad + {}^C\mathcal{D}^\nu e^T(t-\mu) C^T {K_2}^T Q_2 K_2 C\,{}^C\mathcal{D}^\nu e(t-\mu)
\end{aligned} \tag{13}$$

From (5), the following expression holds

$$\begin{aligned}
&2(e^T(t-\mu)\epsilon_1 + {}^C\mathcal{D}^\nu e^T(t-\mu)\epsilon_2 + {}^C\mathcal{D}^\nu e^T(t)\epsilon_3)P \\
&\times \Big(-{}^C\mathcal{D}^\nu e\,(t) + \mathcal{H}g(De\,(t), \eta(t)) \\
&+\mathcal{A}e\,(t) - K_1Ce\,(t-\mu) - K_2C^C\mathcal{D}^\nu e\,(t-\mu)\Big) = 0
\end{aligned} \tag{14}$$

From (7), for any diagonal positive matrix,

$$\begin{aligned}
2\Big[&e^T(t)D^T KTg\,(De\,(t), \eta(t)) \\
&- g^T\,(De\,(t), \eta(t))\,Tg\,(De\,(t), \eta(t))\Big] \ge 0
\end{aligned} \tag{15}$$

holds. In addition, for any real matrices R_{12}, $R_{11} = {R_{11}}^T$ and $R_{22} = {R_{22}}^T$, that satisfy

$$R = \begin{pmatrix} R_{11} & R_{12} \\ * & R_{22} \end{pmatrix} \ge 0 \tag{16}$$

we have

$$\mu^{\nu_2}{\nu_1}^{-1}\Lambda^T(t)R\Lambda(t) - \int_{t-\mu}^{t} (t-\kappa)^{\nu-1}\Lambda^T(t)R\Lambda(t)d\kappa \ge 0 \tag{17}$$

where $\Lambda^T(t) = \left[e^T(t),\ \mathcal{D}^\nu e^T(t)\right]$. To simplify the calculations, let $R_{11} = R_{22} = I, R_{12} = 0$. Integrating (13) with (14), (15) and (17) yields

$$
\begin{aligned}
&{}^C\mathcal{D}^\nu V(t) + e^T(t)\, Q_1 e\,(t)\\
&\quad + \left(K_1 C e\,(t-\mu) + K_2 C\, {}^C\mathcal{D}^\nu e\,(t-\mu)\right)^T\\
&\quad \times Q_2 (K_1 C e\,(t-\mu) + K_2 C\, {}^C\mathcal{D}^\nu e\,(t-\mu))\\
&\le 2e^T(t) P\mathcal{H} g(De\,(t), \eta(t))\\
&\quad + 2e^T(t) P\mathcal{A} e\,(t)\\
&\quad - 2e^T(t) P K_1 C e\,(t-\mu)\\
&\quad - 2e^T(t) P K_2 C\, {}^C\mathcal{D}^\nu e\,(t-\mu)\\
&\quad + e^T(t)\, Q_1 e\,(t)\\
&\quad + e^T(t-\mu)(K_1 C)^T Q_2 (K_1 C)\, e\,(t-\mu)\\
&\quad + e^T(t-\mu)(K_1 C)^T Q_2 (K_2 C)\, {}^C\mathcal{D}^\nu e\,(t-\mu)\\
&\quad + {}^C\mathcal{D}^\nu e^T(t-\mu)\, C^T K_2{}^T Q_2 K_1 C e\,(t-\mu)\\
&\quad + {}^C\mathcal{D}^\nu e^T(t-\mu) C^T K_2{}^T Q_2 K_2 C\, {}^C\mathcal{D}^\nu e(t-\mu)\\
&\quad + 2e^T(t) D^T K T g\,(De\,(t), \eta(t))\\
&\quad - 2g^T\,(De\,(t), \eta(t))\, T g\,(De\,(t), \eta(t))\\
&\quad - 2e^T(t-\mu)\epsilon_1 P\, {}^C\mathcal{D}^\nu e\,(t)\\
&\quad + 2e^T(t-\mu)\epsilon_1 P\mathcal{A} e\,(t)\\
&\quad + 2e^T(t-\mu)\epsilon_1 P\mathcal{H} g(De\,(t), \eta(t))\\
&\quad - 2e^T(t-\mu)\epsilon_1 P K_1 C e\,(t-\mu)\\
&\quad - 2e^T(t-\mu)\epsilon_1 P K_2 C\, {}^C\mathcal{D}^\nu e\,(t-\mu)\\
&\quad - 2\, {}^C\mathcal{D}^\nu e^T(t-\mu)\epsilon_2 P\, {}^C\mathcal{D}^\nu e\,(t)\\
&\quad + 2\, {}^C\mathcal{D}^\nu e^T(t-\mu)\epsilon_2 P\mathcal{A} e\,(t)\\
&\quad + 2\, {}^C\mathcal{D}^\nu e^T(t-\mu)\epsilon_2 P\mathcal{H} g(De\,(t), \eta(t))\\
&\quad - 2\, {}^C\mathcal{D}^\nu e^T(t-\mu)\epsilon_2 P K_1 C e\,(t-\mu)\\
&\quad - 2\, {}^C\mathcal{D}^\nu e^T(t-\mu)\epsilon_2 P K_2 C\, {}^C\mathcal{D}^\nu e\,(t-\mu)\\
&\quad - 2\, {}^C\mathcal{D}^\nu e^T(t)\epsilon_3 P\, {}^C\mathcal{D}^\nu e\,(t)\\
&\quad + 2\, {}^C\mathcal{D}^\nu e^T(t)\epsilon_3 P\mathcal{A} e\,(t)\\
&\quad + 2\, {}^C\mathcal{D}^\nu e^T(t)\epsilon_3 P\mathcal{H} g(De\,(t), \eta(t))\\
&\quad - 2\, {}^C\mathcal{D}^\nu e^T(t)\epsilon_3 P K_1 C e\,(t-\mu)\\
&\quad - 2\, {}^C\mathcal{D}^\nu e^T(t)\epsilon_3 P K_2 C\, {}^C\mathcal{D}^\nu e\,(t-\mu)\\
&\quad + \mu^{\nu_2}\nu_1{}^{-1} e^T(t) e(t)\\
&\quad + \mu^{\nu_2}\nu_1{}^{-1} e^T(t)\mathcal{A}^T\mathcal{A} e(t)\\
&\quad + \mu^{\nu_2}\nu_1{}^{-1} e^T(t)\mathcal{A}^T\mathcal{H} g\,(De(t), \eta(t))\\
&\quad - \mu^{\nu_2}\nu_1{}^{-1} e^T(t)\mathcal{A}^T K_1 C e(t-\mu)\\
&\quad - \mu^{\nu_2}\nu_1{}^{-1} e^T(t)\mathcal{A}^T K_2 C\, {}^C\mathcal{D}^\nu e(t-\mu)\\
&\quad + \mu^{\nu_2}\nu_1{}^{-1} g^T\,(De(t), \eta(t))\, \mathcal{H}^T\mathcal{A} e(t)\\
&\quad + \mu^{\nu_2}\nu_1{}^{-1} g^T\,(De(t), \eta(t))\, \mathcal{H}^T\mathcal{H} g\,(De(t), \eta(t))\\
&\quad - \mu^{\nu_2}\nu_1{}^{-1} g^T\,(De(t), \eta(t))\, \mathcal{H}^T K_1 C e(t-\mu)\\
&\quad - \mu^{\nu_2}\nu_1{}^{-1} g^T\,(De(t), \eta(t))\, \mathcal{H}^T K_2 C\, {}^C\mathcal{D}^\nu e(t-\mu)\\
&\quad - \mu^{\nu_2}\nu_1{}^{-1} e^T(t-\mu)(K_1 C)^T\mathcal{A} e(t)\\
&\quad - \mu^{\nu_2}\nu_1{}^{-1} e^T(t-\mu)(K_1 C)^T\mathcal{H} g\,(De(t), \eta(t))\\
&\quad + \mu^{\nu_2}\nu_1{}^{-1} e^T(t-\mu)(K_1 C)^T K_1 C e(t-\mu)\\
&\quad + \mu^{\nu_2}\nu_1{}^{-1} e^T(t-\mu)(K_1 C)^T K_2 C\, {}^C\mathcal{D}^\nu e(t-\mu)\\
&\quad - \mu^{\nu_2}\nu_1{}^{-1} \mathcal{D}^\nu e^T(t-\mu)(K_2 C)^T\mathcal{A} e(t)\\
&\quad - \mu^{\nu_2}\nu_1{}^{-1} \mathcal{D}^\nu e^T(t-\mu)(K_2 C)^T\mathcal{H} g\,(De(t), \eta(t))\\
&\quad + \mu^{\nu_2}\nu_1{}^{-1} \mathcal{D}^\nu e^T(t-\mu)(K_2 C)^T K_1 C e(t-\mu)\\
&\quad + \mu^{\nu_2}\nu_1{}^{-1} \mathcal{D}^\nu e^T(t-\mu)(K_2 C)^T K_2 C\, {}^C\mathcal{D}^\nu e(t-\mu)\\
&\quad - \int_{t-\mu}^{t} (t-\kappa)^{\nu-1}\Lambda^T(t) R\Lambda(t) d\kappa
\end{aligned} \tag{18}
$$

Applying Lemma 3, it yields

$$
\begin{aligned}
&e^T(t-\mu)(K_1 C)^T Q_2 (K_2 C)\, {}^C\mathcal{D}^\nu e\,(t-\mu)\\
&\quad + {}^C\mathcal{D}^\nu e^T(t-\mu) C^T K_2{}^T Q_2 K_1 C e(t-\mu)\\
&\le e^T(t-\mu) C^T K_1{}^T \lambda K_1 C e(t-\mu)\\
&\quad + {}^C\mathcal{D}^\nu e^T(t-\mu) C^T K_2{}^T Q_2{}^T \lambda^{-1} Q_2 K_2 C\, {}^C\mathcal{D}^\nu e\,(t-\mu)
\end{aligned} \tag{19}
$$

Integrating (19) with (18) yields

$$
\begin{aligned}
&{}^C\mathcal{D}^\nu V(t) + e^T(t)\, Q_1 e\,(t)\\
&\quad + \left(K_1 C e\,(t-\mu) + K_2 C\, {}^C\mathcal{D}^\nu e\,(t-\mu)\right)^T\\
&\quad \times Q_2 (K_1 C e\,(t-\mu) + K_2 C\, {}^C\mathcal{D}^\nu e\,(t-\mu))\\
&\le \varpi^T(t)\, \Omega\varpi\,(t) - \int_{t-\mu}^{t} (t-\kappa)^{\nu-1}\Lambda^T(t) R\Lambda(t) d\kappa
\end{aligned} \tag{20}
$$

where

$$
\begin{aligned}
\varpi^T(t) = \big[\ & e^T(t) \quad e^T(t-\mu) \quad g^T(De\,(t), \eta(t))\\
& {}^C\mathcal{D}^\nu e^T(t) \quad {}^C\mathcal{D}^\nu e^T(t-\mu)\ \big]
\end{aligned} \tag{21}
$$

$$
\Omega = \begin{pmatrix}
\Omega_{11} & \Omega_{12} & \Omega_{13} & \mathcal{A}^T\epsilon_3 P & \Omega_{15}\\
* & \Omega_{22} & \epsilon_1 P\mathcal{H} - \mu^{\nu_2}\nu_1{}^{-1}(K_1 C)^T\mathcal{H} & \Omega_{24} & \Omega_{25}\\
* & * & -2T + \mu^{\nu_2}\nu_1{}^{-1}\mathcal{H}^T\mathcal{H} & \mathcal{H}^T\epsilon_3 P & \mathcal{H}^T\epsilon_2 P - \mu^{\nu_2}\nu_1{}^{-1}\mathcal{H}^T K_2 C\\
* & * & * & -2\epsilon_3 P & -\epsilon_2 P - \epsilon_3 P K_2 C\\
* & * & * & * & \Omega_{55}
\end{pmatrix} < 0 \tag{22}
$$

with $\Omega_{11} = P\mathcal{A} + \mathcal{A}^T P + Q_1 + \mu^{\nu_2}\nu_1{}^{-1} I + \mu^{\nu_2}\nu_1{}^{-1}\mathcal{A}^T\mathcal{A}$, $\Omega_{12} = -PK_1C + \mathcal{A}^T\epsilon_1 P - \mu^{\nu_2}\nu_1{}^{-1}\mathcal{A}^T K_1 C$, $\Omega_{13} = P\mathcal{H} + D^T K T + \mu^{\nu_2}\nu_1{}^{-1}\mathcal{A}^T\mathcal{H}$, $\Omega_{22} = (K_1C)^T Q_2 (K_1 C) + C^T K_1{}^T\lambda K_1 C - \epsilon_1 P K_1 C + \mu^{\nu_2}\nu_1{}^{-1}(K_1C)^T K_1 C$, $\Omega_{24} = -\epsilon_1 P - C^T K_1{}^T\epsilon_3 P$, $\Omega_{25} = -\epsilon_1 P K_2 C - (K_1 C)^T\epsilon_2 P + \mu^{\nu_2}\nu_1{}^{-1}(K_1C)^T K_2 C$, $\Omega_{55} = -\epsilon_2 P K_2 C - C^T K_2{}^T\epsilon_2 P + (K_2C)^T Q_2{}^T\lambda^{-1} Q_2 (K_2 C) + (K_2 C)^T Q_2 (K_2 C) + \mu^{\nu_2}\nu_1{}^{-1}(K_2C)^T K_2 C$.

Using Lemma 2 results in

$$\left(\begin{array}{ccccccccccc}
\Theta_{11} & \Theta_{12} & \Theta_{13} & \Theta_{14} & \Theta_{15} & Q_1 & 0 & 0 & 0 & 0 & \mu^{\nu_2}{\nu_1}^{-1}\mathcal{A}^T \\
* & \Theta_{22} & \epsilon_1 P\mathcal{H} & \Theta_{24} & \Theta_{25} & 0 & (K_1C)^T & (K_1C)^T & 0 & 0 & -\mu^{\nu_2}{\nu_1}^{-1}(K_1C)^T \\
* & * & -2T & \mathcal{H}^T\epsilon_3 P & \mathcal{H}^T\epsilon_2 P & 0 & 0 & 0 & 0 & 0 & \mu^{\nu_2}{\nu_1}^{-1}\mathcal{H}^T \\
* & * & * & -2\epsilon_3 P & \Theta_{45} & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & \Theta_{55} & 0 & 0 & 0 & (K_2C)^T{Q_2}^T & (K_2C)^T & -\mu^{\nu_2}{\nu_1}^{-1}(K_2C)^T \\
* & * & * & * & * & -Q_1 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -{Q_2}^{-1} & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & -\lambda^{-1} & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & -\lambda & 0 & 0 \\
* & * & * & * & * & * & * & * & * & -{Q_2}^{-1} & 0 \\
* & * & * & * & * & * & * & * & * & * & -\mu^{\nu_2}{\nu_1}^{-1}I
\end{array}\right) \tag{23}$$

where $\Theta_{11} = P\mathcal{A} + \mathcal{A}^T P + \mu^{\nu_2}{\nu_1}^{-1}I$, $\Theta_{12} = -PK_1C + \mathcal{A}^T\epsilon_1 P$, $\Theta_{13} = P\mathcal{H} + D^T KT$, $\Theta_{14} = \mathcal{A}^T\epsilon_3 P$, $\Theta_{15} = -PK_2C + \mathcal{A}^T\epsilon_2 P$, $\Theta_{22} = -\epsilon_1 PK_1C - (K_1C)^T\epsilon_1 P$, $\Theta_{24} = -\epsilon_1 P - C^T {K_1}^T\epsilon_3 P$, $\Theta_{25} = -\epsilon_1 PK_2C - (K_1C)^T\epsilon_2 P$, $\Theta_{45} = -\epsilon_2 P - \epsilon_3 PK_2C$, and $\Theta_{55} = -\epsilon_2 PK_2C - (K_2C)^T\epsilon_2 P$. Multiplying (23) from the left and right by $\mathrm{diag}\{P^{-1}, P^{-1}, T^{-1}, P^{-1}, P^{-1}, I, I, I, I, I, I\}$, and defining $\bar{P} = P^{-1}$, $\bar{T} = T^{-1}$, $X = K_1CP^{-1}$, and $Z = K_2CP^{-1}$. Then, using Lemma 2, the condition (9) is established. Applying Lemma 4 ensures asymptotic stability of the error system (5). Finally, combining (20), (22), and the control law (3) yields the guaranteed cost bound.

$$\begin{aligned}
{}^{C}\mathcal{D}^{\nu}V(t) &\leq \varpi^T(t)\,\Omega\varpi(t) - e^T(t)\,Q_1 e(t) \\
&\quad - u^T(t)Q_2 u(t) \\
&\leq -(u^T(t)Q_2 u(t) + e^T(t)\,Q_1 e(t)) \\
&\leq 0
\end{aligned} \tag{24}$$

where

$$\begin{aligned}
u^T(t)Q_2 u(t) =&(K_1Ce(t-\mu) + K_2C\,{}^{C}\mathcal{D}^{\nu}e(t-\mu))^T \\
&\times Q_2(K_1Ce(t-\mu) + K_2C\,{}^{C}\mathcal{D}^{\nu}e(t-\mu))
\end{aligned}$$

Integrating both sides of (24) with order ν results

$$V(\mathfrak{h}, e(\mathfrak{h})) - V(0, e(0)) \leq -J(u) \tag{25}$$

As $V(\mathfrak{h}, e(\mathfrak{h})) \geq 0$, we get

$$\begin{aligned}
J(u) &\leq V(0, e(0)) - V(h, e(\mathfrak{h})) \\
&\leq V(0, e(0)) \\
&\leq \lambda_{\max}(\bar{P}^{-1})\|\psi\|^2 = J^*
\end{aligned} \tag{26}$$

V Simulations

The Chua's circuit is a special case of Lur'e systems [57], as it consists of a linear part and a sector-bounded nonlinearity, a structure shared by many chaotic systems, which is modeled by the following VFO state-space form for simulation purposes

$$\begin{aligned}
{}^{C}\mathcal{D}^{\nu}\eta_1 &= a(\eta_2 - \eta_1 - g(\eta_1)) \\
{}^{C}\mathcal{D}^{\nu}\eta_2 &= \eta_1 - \eta_2 + \eta_3 \\
{}^{C}\mathcal{D}^{\nu}\eta_3 &= -b\eta_2
\end{aligned} \tag{27}$$

Consider the state vectors η_1, η_2, and η_3 whose components are defined as the capacitor voltages and the current through the inductor, respectively. $g(\eta_1)$ stands for the nonlinear characteristic of Chua's diode, defined explicitly as:

$$g(\eta_1) = m_1\eta_1 + \frac{1}{2}(m_0 - m_1)\ \ (|\eta_1 + 1| - |\eta_1 - 1|)\,. \tag{28}$$

(27) is equivalent to the VFOLSs from with

$$\mathcal{H} = \begin{bmatrix} -a(m_0 - m_1) \\ 0 \\ 0 \end{bmatrix},\ \mathcal{A} = \begin{bmatrix} -a\,m_1 & a & 0 \\ 1 & -1 & 1 \\ 0 & -b & 0 \end{bmatrix}, \tag{29}$$

$$D = C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

with $m_1 = -2m_0 = 0.285$, $b = 14.28$, $a = 9$, and $\mathcal{F}(\eta_1(t)) = (1/2)\,(|\eta_1 + 1| - |\eta_1 - 1|)$ belongs to the sector $[0, 1]$. First, to evaluate the effectiveness of the proposed guaranteed cost PD controller, we apply the design procedure described in Theorem. we set $\epsilon_1 = 0.01$, $\epsilon_2 = \epsilon_3 = 0.02$. Setting $Q_2 = 0.5, Q_1 = I$ and $\nu = 0.1\,|\tanh(t)| + 0.8$ that is, $\nu_1 = 0.8$, $\nu_2 = 0.1$. Using the parameters $\mu = 0.2$ and $\lambda = 1$, we obtain

$$\bar{P} = \begin{pmatrix} 0.4946 & -0.0051 & -0.0449 \\ -0.0051 & 0.0374 & 0.0329 \\ -0.0449 & 0.0329 & 0.4926 \end{pmatrix}$$

$$X = \begin{pmatrix} 0.0125 & -0.0015 & -0.0027 \\ -0.0027 & 0.0053 & -0.0033 \\ 0.0002 & 0.0031 & 0.0058 \end{pmatrix}$$

$$Z = \begin{pmatrix} 0.0095 & 0.0001 & -0.0025 \\ -0.0011 & 0.0069 & -0.0054 \\ 0.0012 & 0.0027 & 0.0057 \end{pmatrix}, \bar{T} = 0.1354$$

The proposed design procedure yields the gain matrices

$$\begin{aligned}
K_1 &= \begin{bmatrix} 0.0247 & -0.0054 & 0.0019 \end{bmatrix}^T \\
K_2 &= \begin{bmatrix} 0.0190 & -0.0023 & 0.0039 \end{bmatrix}^T
\end{aligned}$$

The distinct initial conditions for the master and slave systems are chosen as $\eta(0) = \begin{bmatrix} 0.2 & -0.3 & 0.2 \end{bmatrix}^T$ and $Y(0) = \begin{bmatrix} 0.4 & 0.1 & 0.6 \end{bmatrix}^T$, respectively. The actual cost achieved by the proposed controller is computed using the cost functional in Eq. (8). For this example, the achieved cost is $J = 13.5928$, while the guaranteed bound is $J^* = 28.5496$, confirming that $J < J^*$. The results are presented in Figs. 1–4. Figure 1 depicts the master and slave state trajectories, where it is clearly observed that the slave states converge to the master states over time, confirming the effectiveness of the proposed synchronization scheme. Figure 2 illustrates the chaotic attractors of master and slave systems, respectively, demonstrating that both systems exhibit similar chaotic behavior and that the slave system successfully tracks the master dynamics. The synchronization error resulting from these gains is depicted in Fig. 3, which clearly shows that the error converges to zero asymptotically. This confirms that master-slave synchronization is successfully achieved. Figure 4 illustrates the control input signals confirming the practical applicability of the proposed controller. Next, to demonstrate the superiority and effectiveness of the proposed guaranteed cost PD controller, we compared our approach with a relevant existing method from the literature [29]. The comparison was performed under identical conditions, including the same system parameters and initial conditions. Notice that our method handles VFO $\nu(t)$, whereas the existing method assumes a constant FO. Despite this additional complexity, and despite the presence of input delay, our method achieves a settling time of 8 seconds compared to 10 seconds for the existing method. These results are illustrated in Fig. 5, confirm the superiority of the proposed method, even under more challenging conditions.

VI Conclusion

The synchronization of VFOLSs in a master-slave configuration has been thoroughly investigated using a delayed PD feedback controller. The guaranteed cost control technique has been employed to derive synchronization conditions for the entire system. To validate the theoretical results, the design has been successfully applied to a representative example, namely the VFO Chua's circuit. It should be emphasized that the proposed control design is developed for nominal VFOLSs without parametric uncertainties. The robustness analysis against uncertainties and the determination of the maximum allowable uncertainty are not addressed in this paper and remain as potential directions for future research.

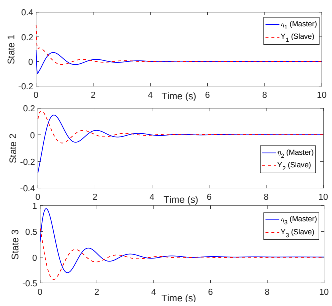
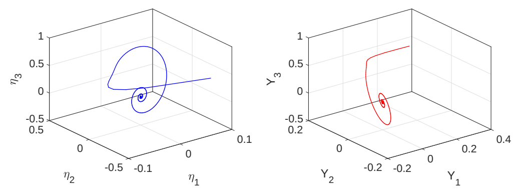


Figure 1. Master and slave state trajectories.



(a) (b)

Figure 2. Chaotic attractors of master and slave systems.

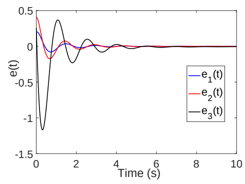


Figure 3. Synchronization errors.

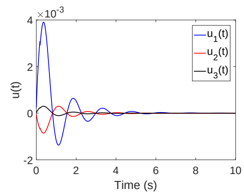


Figure 4. Control input signals.

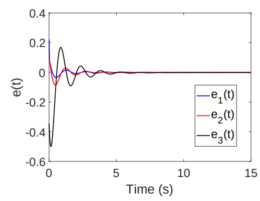


Figure 5. Synchronization errors for the proposed controller.

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